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A DYNAMIC SYSTEM WITH A DISCONTINUOUS CHARACTERISTIC

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We consider a differential equation (which finds practical applications), containing a function with finite discontinuities, and its derivative. Since the equation is not defined on the lines of discontinuity, additional definitions are constructed for various cases.

The supplementary definition schemes supply the information needed for a complete qualitative analysis of the system. Examples of such analysis are given for two different characteristics corresponding to two differing cases of supplementary definition.

Equation
$$\varphi^{\cdot\cdot} + 2h[1 - bF'(\varphi)]\ \varphi^{\cdot} + F(\varphi) = \Omega, \quad F(\varphi + 2\pi) = F(\varphi) \tag{1}$$

encountered in practice (*) was studied often (see e.g. [1] and [2]) for the case when $F(\varphi)$ has continuous characteristics.

Here we propose a method of investigation of (1), when the characteristic exhibits finite discontinuities.

Let $\varphi = \varphi_n$ be one of the points of discontinuity. The system
$$\varphi^{\cdot} = y, \quad y^{\cdot} = \Omega - F(\varphi) - 2h[1 - bF'(\varphi)]y \tag{2}$$
equivalent to (1) is not defined on the line $\varphi = \varphi_n$. Therefore, when the representative

*) When $b > 0$, Eq. (1) represents the equation of the phase automatic frequency control (afc) with an integrating filter with delay; when $b < 0$, Eq. (1) is the equation of automatic control with a proportional integrating filter without delay [1].

point of the system arrives at this line, its motion is no longer known and must, therefore, be defined additionally. We shall do it as follows. Within the interval $(\varphi_n - \mu, \varphi_n + \mu)$ we shall replace the characteristic $F(\varphi)$ with a straight line connecting the points $[\varphi_n - \mu, F(\varphi_n + 0)]$ and $[\varphi_n + \mu, F(\varphi_n - 0)]$. Then the system (2) will be replaced, on the phase plane within the strip $\varphi_n - \mu < \varphi < \varphi_n + \mu$, by

$$x^{\cdot} = y, \qquad y^{\cdot} = \frac{\beta - \gamma}{2\mu} x - 2\alpha y - \frac{\beta + \gamma}{2} \tag{3}$$

where

$$x = \varphi - \varphi_n, \quad \beta = F(\varphi_n - 0) - \Omega, \quad \gamma = F(\varphi_n + 0) - \Omega,$$

$$\alpha = h \quad \left(1 + b\,\frac{\beta - \gamma}{2\mu}\right)$$

Let us now assume that a point whose ordinate is y_0, is fixed on one of the straight lines $x = \pm\mu$. We may find that the half-trajectory $x = x(t, \mu)$, $y = y(t, \mu)$ of the system (3) originating at this point at the time $t = 0$ and passing into the strip at all, sufficiently small μ, may leave the strip at some $t = t(\mu)$ at a point whose coordinates are $x = \mu$, $y = y(\mu)$ $(x = -\mu, y = y(\mu))$. In this case we shall use the following supplementary definition: having reached the point $\varphi = \varphi_n$, $y = y_0$, the representative point of the system (2) remains on the line $\varphi = \varphi_n$ for a period of time t equal to its limiting value $t(\mu)$ as $\mu \to 0$, after which it continues the motion in accordance with (2), in $\varphi > \varphi_n$ $(\varphi < \varphi_n)$ with initial conditions $\varphi = \varphi_n$ and $y = \lim y(\mu)$ as $\mu \to 0$. If, on the other hand, the half-trajectory of (3) which we are now discussing lies completely within the strip, at all sufficiently small μ, then we assume that, having reached the point $\varphi = \varphi_n$, $y = y_0$, the representative point of (2) remains on the straight line indefinitely. These two possibilities meet all the situations encountered in practice. Equation of motion of the representative point along the line $\varphi = \varphi_n$ can in both cases be obtained by passing to the limit in the equations of the considered half-trajectory of the system (3): $y = Y(t) = \lim y(t, \mu)$ as $\mu \to 0$.

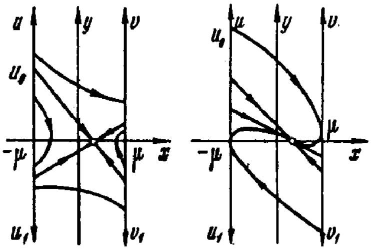
Let us now perform some calculations using the concepts of point transformations [3]. Let us define (Fig. 1) on the line $x = -\mu$, two half-lines

$$U\,\{x = -\mu,\ y = u > 0\}, \qquad U_1\,\{x = -\mu,\ y = -u_1 < 0\}$$

and on the line $x = \mu$,

$$V\,\{x = \mu,\ y = v > 0\}, \qquad V_1\,\{x = \mu,\ y = -v_1 < 0\}$$

Trajectories of the system (3) execute point transformations of the half-line U into the half-lines V and U_1. We shall call these transformations T and S, respectively.



a) Fig. 1 b)

Then

$$u = \frac{2\omega\mu}{\beta - \gamma}\left[-\gamma\,\frac{\exp \alpha\tau}{\operatorname{sh} \omega\tau} + \beta\left(\operatorname{cth} \omega\tau + \frac{\alpha}{\omega}\right)\right]$$

$$v = \frac{2\omega\mu}{\beta - \gamma}\left[\beta\,\frac{\exp(-\alpha\tau)}{\operatorname{sh} \omega\tau} - \gamma\left(\operatorname{cth} \omega\tau - \frac{\alpha}{\omega}\right)\right] \tag{4}$$

gives the parametric equations of the mapping function for the transformation T, and

$$u = \frac{2\omega\mu\beta}{\beta - \gamma}\left[\operatorname{cth} \omega\theta + \frac{\alpha}{\omega} - \frac{\exp \alpha\theta}{\operatorname{sh} \omega\theta}\right] \qquad u_1 = \frac{2\omega\mu\beta}{\beta - \gamma}\left[\operatorname{cth} \omega\theta - \frac{\alpha}{\omega} - \frac{\exp(-\alpha\theta)}{\operatorname{sh} \omega\theta}\right]$$

$$\left(\omega = \sqrt{\alpha^2 + (\beta - \gamma)/2\mu}\right) \tag{5}$$

for the transformation S, where τ and θ are the parameters (denoting the times of transition of the representative point of (3) from U to V or U_1, respectively).

We shall consider the following cases.

1) $\gamma < 0 < \beta$, $r = 2b$, $h > 0$. Saddle type equilibrium state of (3) is situated within the strip $-\mu < x < \mu$. The corresponding pattern of phase trajectories is shown on Fig.1a.

Let u_0 be a segment cut off the half line by the separatrix with a negative slope. The quantity $u_0 \to r\beta$ when $\mu \to 0$. Consequently, when μ is sufficiently small, then any fixed point u lying in the interval $u > r\beta$ will take part in the transformation T. At the same time the first equation of (4) defines the function $\tau = \tau(\mu)$ implicitly, i.e. the time of transition from the fixed u to the half-line V. Its limiting value when $\mu \to 0$ will be (see Appendix 1)

$$\tau = \begin{cases} r \ln [r\gamma / (r\beta - u)] & (r\beta < u < r(\beta - \gamma)) \\ 0 & (u \geqslant r(\beta - \gamma)) \end{cases} \tag{6}$$

Putting $\tau = \tau(\mu)$ in the second equation of (4) we find the $\lim v[\tau(\mu), \mu]$ as $\mu \to 0$ (see Appendix 2), which is

$$v = \begin{cases} 0 & (r\beta < u < r(\beta - \gamma)) \\ u - r(\beta - \gamma) & (u \geqslant r(\beta - \gamma)) \end{cases} \tag{7}$$

Let us now fix an arbitrary point on the interval $u < r\beta$. At small μ, this point will participate in the transformation S. At the same time the first equation of (5) gives implicitly the function $\theta = \theta(\mu)$, i.e. the time of transition from the fixed u to the half-line U_1. Its limiting value as $\mu \to 0$ is

$$\theta = r \ln[r\beta / (r\beta - u)] \tag{8}$$

(the proof is analogous to that in Appendix 1).

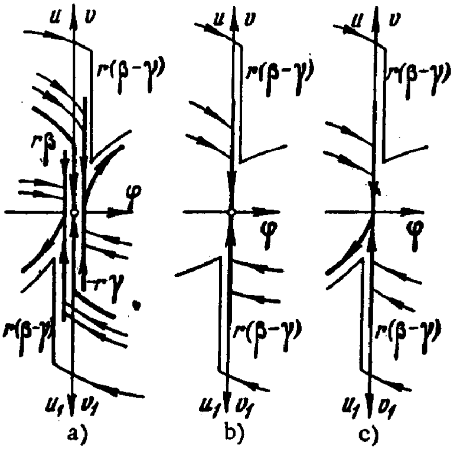
Putting $\theta = \theta(\mu)$ in the second equation of (5) we find the $\lim u_1[\theta(\mu), \mu]$ as $\mu \to 0$, which is

$$u_1 = 0 \tag{9}$$

Let us now suppose that $u = r\beta$ is fixed, and see how the segment $u_0 = u_0(\mu)$ varies with decreasing μ. To do this, we shall compute

$$u_0'(0) = 2\beta(4h^2b + 1) / r(\beta - \gamma)$$

If $b > -1/4h^2$, then, for small μ, the value of u_0 decreases together with μ, and the point $u = r\beta$ participates in the transformation S. First equation of (5) defines the function $\theta = \theta(\mu)$ which defines the time of transition from $u = r\beta$ to the half-line U_1. Its limiting value as $\mu \to 0$, is $\theta = \infty$.



a) b) c)

Fig. 2

This follows from the fact that θ in (8) can be made arbitrarily large by choosing u sufficiently near to $r\beta$ and from the fact that $\partial u / \partial\theta > 0$ in (5).

The case $b < -1/4h^2$ can be treated in the similar manner. The quantity u_0 increases with decreasing μ, and the point $u = r\beta$ participates in the transformation T, the limiting time is $\tau = \infty$.

When $b = -1/4h^2$, we have $r\beta \equiv u_0$, the point $u = r\beta$ lies on the separatrix at any value of μ and the time of motion requiring additional definition, is infinite.

When $u < r(\beta - \gamma)$ we require, in addition, an equation of motion of the representative point along the straight line $\varphi = \varphi_n$. This equation will be, for any u,

$$y = Y(t) \equiv 0 \tag{10}$$

Fig. 2a shows the resulting scheme of additionally defined motions on the upper part of the straight line $\varphi = \varphi_n$. To make the picture clearer, we have spread the overlapping trajectories in the horizontal direction. On reaching the point $\varphi = \varphi_n$, $y = u$, $u \geqslant r(\beta - \gamma)$, the representative point of (2) effects an instantaneous jump as defined by (7) and continues its motion in $\varphi > \varphi_n$ in accordance with the system (2).

If $r\beta < u < r(\beta - \gamma)$, then the representative point jumps instantaneously to the point $v = 0$, remains there for the period of time defined by (6), and then resumes the motion in $\varphi > \varphi_n$.

When $u = r\beta$, the representative point jumps instantaneously to the point $\varphi = \varphi_n$, $y = 0$ and remains there indefinitely. This is what the motion along the separatrix of the saddle point of (3) degenerates to, when $\mu \to 0$. We shall say that this motion proceeds along the separatrix of the equilibrium state $\varphi = \varphi_n$, $y = 0$.

If $u < r\beta$, then the representative point jumps instantaneously to the point $u_1 = 0$, remains there for the period of time given by (8), and resumes the motion in $\varphi < \varphi_n$ (*).

Symmetry considerations imply that, if we replace in the above scheme u by v_1, v by u_1, β by γ and vice versa, then the resulting scheme of the additionally defined motions will apply to the lower part of the straight line $\varphi = \varphi_n$.

2) $\beta < 0 < \gamma$, $r < 0$. The equilibrium of (3) at small μ is a stable node lying on the strip $-\mu < x < \mu$. Fig. 1b shows the corresponding pattern of phase trajectories.

Let $u_0 = u_0(\mu)$ be a segment cut off the half-line U by a trajectory passing through the point $x = \mu$, $y = 0$. When $\mu \to 0$ (see Appendix 3), $u_0(\mu) \to r(\beta - \gamma)$ and $u_0'(\mu) \to +\infty$. Therefore when $u > r(\beta - \gamma)$ the calculations coincide with the corresponding calculations performed in Section 1. If, on the other hand, the half-trajectory of the system (3) originates at the segment $u \leqslant r(\beta - \gamma)$, then it lies, for small μ, completely within the strip $-\mu < x < \mu$, and the time of the corresponding additional motions is ∞. The equation of motion along the straight line $\varphi = \varphi_n$ is Eq. (10) just as in the case (1).

The pattern of motions on the upper part of the straight line $\varphi = \varphi_n$ defined additionally, is shown on Fig. 2b. On reaching the point $\varphi = \varphi_n$, $y = u$, $u > r(\beta - \gamma)$, the representative point of the system (2) effects an instantaneous jump as defined by (7) and continues to move in $\varphi > \varphi_n$ in accordance with the system (2).

When $u \leqslant r(\beta - \gamma)$, the representative point jumps instantaneously to the point $\varphi = \varphi_n$, $y = 0$ and remains there indefinitely. We shall say that these motions tend to the state of equilibrium $\varphi = \varphi_n$, $y = 0$, which is a degenerate form of the node of (3) when $\mu \to 0$.

Symmetry considerations imply, that, if we replace in the system under discussion u by v_1 and v by u_1, then the resulting system of motions will take place on the lower

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*) Equation (1) can be rewritten as

$$\varphi'' + 2h\varphi' = rz' - z, \qquad z = F(\varphi) - \Omega$$

and the Aizerman and Gantmakher [4] discontinuity conditions applied.

When $u > r(\beta - \gamma)$ we obtain the power line of (7). The discontinuity conditions cannot however be applied when $u < r(\beta - \gamma)$.

part of the straight line $\varphi = \varphi_n$.

3) $0 < \gamma < \beta$, $r > 0$. System (3) has a saddle type equilibrium situated to the right of the strip $-\mu < x < \mu$.

We show the resulting pattern of motions on Fig. 2c, while omitting the relevant computations. On reaching the point $\varphi = \varphi_n$, $y = u$, $u > r(\beta - \gamma)$ the representative point of (2) effects an instantaneous jump as defined by (7) and continues the motion in $\varphi > \varphi_n$ in accordance with the system (2).

If $u \leqslant r(\beta - \gamma)$, then the representative point jumps instantaneously to $u_1 = 0$ remains there for the period of time defined by (8) and continues its motion in $\varphi < \varphi_n$. When the representative point of (2) reaches the point $\varphi = \varphi_n$, $y = v_1$, $v_1 \geqslant r(\beta - \gamma)$, it effects an instantaneous upward jump in accordance with the equation $u_1 = v_1 \geqslant r(\beta - \gamma)$ and continues its motion in $\varphi < \varphi_n$ in accordance with the system (2).

If $v_1 < r(\beta - \gamma)$, then the representative point jumps instantaneously to the point $u_1 = 0$ and remains there for the period of time given by $\tau = r \ln [r\beta / (r\gamma + v_1)]$, after which it continues its motion in $\varphi < \varphi_n$.

4) $0 < \beta < \gamma$, $r < 0$. At small μ the system (3) has a stable equilibrium in the form of a node situated to the left of the strip $-\mu < x < \mu$. The scheme of supplementary motions is identical to that of the case (3).

Cases (5) $\beta < \gamma < 0$, $r < 0$ and (6) $\gamma < \beta < 0$, $r > 0$ can be obtained from the cases (4) and (3), respectively, by replacing β with $-\gamma$, u with v_1, v with u_1 and vice versa.

The above six cases yield six more, corresponding to the change of the sign of r, replacement of β with $-\gamma$, r with $-r$, u with v, u_1 with v_1 and vice versa.

Thus, considering the point transformations of the straight lines $\varphi = \varphi_n$ into themselves and one into the other effected by the trajectories of the system (2), we can perform the qualitative analysis of the system for some specific characteristic $F(\varphi)$ with discontinuities.

Example 1. [5]. Let $F(\varphi)$ be a 2π-periodic function such that $F(\varphi) = \varphi / \pi$ when $-\pi < \varphi < \pi$ and $0 < \Omega < 1$. This example illustrates the first case of supplementary definition.

We shall take the strip bounded by two straight lines $\varphi = -\pi$ and $\varphi = \pi$ as the phase space. By making the points on these lines coincide for the same values of ordinates we find, that the point of discontinuity of the characteristic is situated at the seam of the cylindrical phase space.

Next we shall ascertain the existence of a limit cycle encompassing the phase cylinder and situated within its upper part $y > 0$. To do this, we shall select on the straight line $\varphi = -\pi$ a half-line $V\{\varphi = -\pi,\ y = v > 0\}$, and on the line $\varphi = \pi$, a half-line $U\{\varphi = \pi,\ y = u > 0\}$ and consider a point transformation of the half-line V into the half-line U effected by the trajectories of (2). Putting

$$h_1 = h(\pi - b) / \sqrt{\pi}, \qquad \omega_1 = \sqrt{1 - h_1^2}, \qquad k_1 = h_1 / \omega_1$$

we find, that, when $0 < h_1 < 1$, then the parametric equations of the mapping function of this transformation will be

$$v / \sqrt{\pi} = \omega_1(1 - \Omega) \exp(k_1\eta) / \sin\eta + (1 + \Omega)(\omega_1 \operatorname{ctg} \eta + h_1)$$
$$u / \sqrt{\pi} = \omega_1(1 + \Omega) \exp(-k_1\eta) / \sin\eta + (1 - \Omega)(\omega_1 \operatorname{ctg} \eta - h_1)$$

its derivatives

$$dv/du = \exp\,(2k_1\eta)\; u\,/\,v$$
$$d^2v\,/\,du^2 = \sqrt{\pi}\sin\eta\,\exp\,(3\,k_1\eta)\,[(1-\Omega)v-(1+\Omega)\,u]\,/\,\omega_1 v^3$$

and its asymptotic behavior will be given by

$$v = u + 4\sqrt{\pi}h_1$$

Fig. 3 shows the curve $u = u(v)$ for $a = \ln\,[(1+\Omega)\,/\,(1-\Omega)] > k_1\pi$, together with the broken line

$$v = v\,(u) = \begin{cases} u-2r & \text{when } u \geqslant 2r \\ 0 & \text{when } r\,(1-\Omega) < u < 2r \end{cases}$$

the latter representing the mapping function of the transformation of the half-line $\varphi = \pi,\ y > 0$ into itself, effected by the supplementary motions and constructed according to Formula (7). Fig. 3 illustrates the case when the curve and the broken line have a common point on the horizontal part of the broken line. This means that a limit cycle exists on the phase cylinder, and the representative point moving along this limit cycle will come to rest at the point $\varphi = \pi,\ y = 0$ for the period of time defined by the first line of Formula (6). We easily see that this is the only possible cycle and, that it is stable. When r increases, the point of intersection of the curve with broken line approaches the initial point of the latter, and this corresponds to the merging of the cycle into the separatrix forming a loop (the separatrix emerging from the state of equilibrium $(\pi,\ 0)$ and returning to it). The corresponding bifurcation surface in the parametric (r, h_1, a)-space is defined by the equations $v(\eta) = 0$.

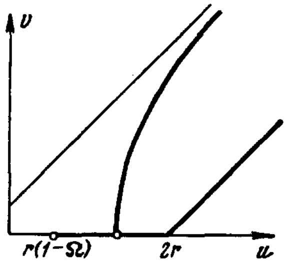

Fig. 3

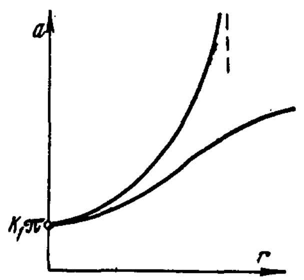

Fig. 4

Eliminating η, we obtain

$$a = k_1\left(\pi - \operatorname{arctg}\frac{\omega_1 r}{\sqrt{\pi}+h_1 r}\right) + \frac{1}{2}\ln\frac{r^2+2\sqrt{\pi}h_1 r+\pi}{\pi}$$

When r decreases, the point of intersection of the curve with the broken line passes to the inclined part of the latter. This corresponds to the transformation of the cycle "stopping" at the point $(\pi,\ 0)$, to the cycle without a "stop". The surface separating, in the parametric space, the domain of existence of the cycle with a stop from that of the cycle without a stop, is given by

$$u(\eta) = 2r,\ v\,(\eta) = 0$$

These equations can be rewritten as

$$r = \sqrt{\pi}\sin\eta\,/\,[\omega_1\exp\,(k_1\eta) - \omega_1\cos\eta - h_1]$$
$$a = k_1\eta - \ln\,(-\cos\eta - k_1\sin\eta)$$

and regarded as the parametric equations of the intersection of the surface with the plane $h_1 = \text{const}$.

Fig. 4 shows how the plane $h_1 = \text{const}$ intersects the parametric space. If the parameter values correspond to the region below the lower curve then the system has no cycles, the region between the curves corresponds to the cycle with a stop, and the region above the upper curve corresponds to a cycle without a stop.

Continuing the investigation we find, that, for a selected region of the parametric space defined by $0 < h_1 < 1$, $a > k_1 \pi$, $r > 0$, the system has no cycles enveloping the cylinder when $y < 0$, nor any cycles not enveloping the cylinder. Qualitative representation of the phase trajectories of the system for the case of a cycle with a stop, is given on Fig. 5.

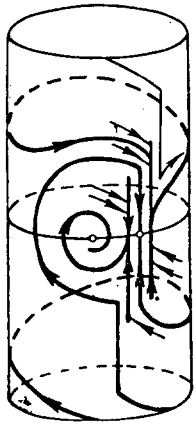

Fig. 5

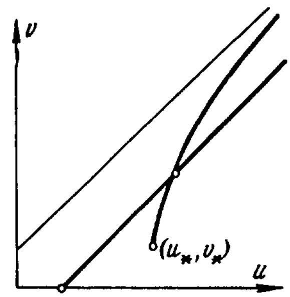

Fig. 6

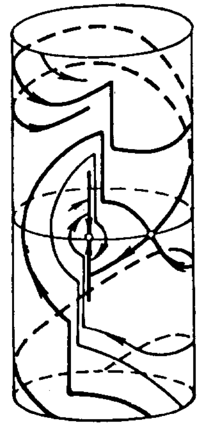

Fig. 7

Example 2. This differs from Example 1 in the signs of $F(\varphi)$ and r and illustrates the second case of supplementary definition.

Equations of the mapping function $u = u(v)$, its derivatives and its asymptotic behavior can be obtained from the corresponding equations of Example 1, by putting $h_1 = h(\pi + \eta b)/\sqrt{\pi}$, $w_1 = \sqrt{1 + h_1^2}$, replacing Ω with $-\Omega$, trigonometric functions with the corresponding hyperbolic functions and putting a minus sign before the second order derivative.

To obtain all the values of v and u, we must vary η from 0 to ∞, v from ∞ to $v_* = \sqrt{\pi}(1 - \Omega)(\omega_1 + h_1)$, and u from ∞ to $u_* = \sqrt{\pi}(1 + \Omega)(\omega_1 - h_1)$, where v_* and u_* denote the segments cut off from the half-lines V and U respectively by the separatrices of the saddle type equilibrium $(-\pi\Omega, 0)$. The curve $u = u(v)$ is shown on Fig. 6, together with the straight line $v = u + 2r$, representing the mapping function of the half-line $\varphi = \pi$, $y > 0$ into itself effected by the additionally defined motions.

Fig. 6 illustrates the case when the curve and the straight line intersect. This means that a limit cycle exists on the phase cylinder. This cycle is the only one possible and it is stable. When r decreases, the straight line moves from the left to right and, at some r, the point of intersection reaches the point (u_*, v_*). At the same time the limit cycle merges into the saddle separatrix $(-\pi\Omega, 0)$ forming a loop. The corresponding bifurcation surface in the parametric (r, h_1, Ω)-space is given by

$$v_* = u_* + 2r, \quad \text{or} \quad \Omega = -r/(\sqrt{\pi}\omega_1) + h_1/\omega_1$$

This is a ruled surface formed by straight lines parallel to the plane $h_1 = 0$. The region of the parameter values satisfying the inequality

$$\Omega > -r/(\sqrt{\pi}\omega_1) + h_1/\omega_1$$

corresponds to the case, when a cycle is present.

Moreover we see, that there are no cycles enveloping the cylinder when $y < 0$, and there are no cycles not enveloping the cylinder in the parametric region defined by $h_1 > 0$, $r < 0$, $0 < \Omega < 1$. Fig. 7 gives the qualitative pattern of phase trajectories for the case when a limit cycle exists.

Appendix 1. Let $r\beta < u < r(\beta - \gamma)$. Since the function $u = u(\tau, \mu)$, given by the first equation of (4) does not exist at $\mu = 0$, we shall supplement its definition by its limit value as $\mu \to 0$. The resulting function

$$f(\tau, \mu) = \begin{cases} u(\tau, \mu) & \text{when } \mu > 0 \\ r[\beta - \gamma \exp(-\tau / r)] & \text{when } \mu = 0 \end{cases}$$

satisfies the following three conditions: when it is continuous; if $\tau_0 = r \ln[r\gamma / (r\beta - u)]$, then $f(\tau_0, 0) = u$; and when its derivative $\partial f / \partial \tau < 0$.

Consequently, with u fixed on the interval under consideration, Eq. $u = f(\tau, \mu)$ defines, in some vicinity of the point $(\tau_0, 0)$, a single-valued continuous function $\tau = \tau_*(\mu)$ such, that $\tau_*(0) = \tau_0$. Since $\tau(\mu) \equiv \tau_*(\mu)$ for $\mu \neq 0$, then $\lim \tau(\mu) = \tau_0$ for $\mu \to 0$.

Let us now suppose that $u \geqslant r(\beta - \gamma)$. Since $\partial u / \partial \tau < 0$, the function $\tau = \tau(\mu)$, corresponding to the fixed u on the interval considerd, is bounded from above by the function $\tau = \tau(\mu)$ corresponding to any u fixed on the interval $r\beta < u < r(\beta - \gamma)$, and this boundary function can be made as nearly equal to zero when $\mu \to 0$ as required, by choosing u sufficiently near to $r(\beta - \gamma)$.

Appendix 2. When $r\beta < u < r(\beta - \gamma)$, this limit value can be found by direct substitution of the limit value of $\tau(\mu)$. If $u \geqslant r(\beta - \gamma)$, then we make the substitution $\varkappa = \omega\tau$ in (4). When u is fixed and μ are small, then the first equation of (4) defines the function $\varkappa = \varkappa(\mu)$ such, that $\lim\limits_{\mu \to 0} \varkappa(\mu) = \frac{1}{2} \ln \frac{u}{u - r(\beta - \gamma)}$

(the proof is analogous to that, given in Appendix 1). Putting now $\varkappa = \varkappa(\mu)$ in the second equation of (4) we find, that, as $\mu \to 0$, $\lim v[\varkappa(\mu), \mu] = u - r(\beta - \gamma)$.

Appendix 3. We shall consider the function $u_0 = u[\tau(\mu), \mu]$, where $\tau = \tau(\mu)$ is a function defined by Eq. $v(\tau, \mu) = 0$. Since $\lim v(\tau, \mu) = -0$ when $\mu \to 0$ and $\lim v(\tau, \mu) = +\infty$, when $\tau \to 0$, it follows that $\tau(\mu) \to 0$ as $\mu \to 0$. Let us put $\varkappa = \omega\tau$. Then $u_0 = u[\varkappa(\mu), \mu]$, where $\varkappa = \varkappa(\mu)$ is a function defined by the equation $v(\varkappa, \mu) = 0$. Since $\lim v(\varkappa, \mu) = r(\beta - \gamma) / (\exp 2\varkappa - 1) > 0$ when $\mu \to 0$ and $\lim v(\varkappa, \mu) = -2\omega\mu (1 - \alpha / \omega) / (\beta - \gamma) < 0$ when $\varkappa \to \infty$, it follows that $\varkappa(\mu) \to \infty$ as $\mu \to 0$. Therefore

$$\lim_{\mu \to 0} \frac{\exp \alpha\tau(\mu)}{\operatorname{sh} \omega\tau(\mu)} = \lim_{\mu \to 0} \frac{2}{\exp[(\omega - \alpha)\tau(\mu)] - \exp[-(1 + \alpha / \omega)\varkappa(\mu)]} = 2$$

$$\left(\lim_{\mu \to 0} (\omega - \alpha) = 1 / r, \quad \lim_{\mu \to 0} \alpha / \omega = 1\right)$$

$$\lim_{\mu \to 0} u_0 = \frac{2}{\beta - \gamma} \lim_{\mu \to 0} (\omega\mu) \left[-\gamma \lim_{\mu \to 0} \frac{\exp \alpha\tau(\mu)}{\operatorname{sh} \omega\tau(\mu)} + \beta \left(\lim_{\varkappa \to \infty} \operatorname{cth} \varkappa + \lim_{\mu \to 0} \frac{\alpha}{\omega}\right)\right] = r(\beta - \gamma)$$

Since

$$\frac{\partial u}{\partial \varkappa} = -\frac{v}{\operatorname{sh} \varkappa} \exp\left(\frac{\alpha\varkappa}{\omega}\right)$$

and when $u = u_0$ then $v = 0$, we have

$$\frac{du_0}{d\mu} = \frac{\partial u_0}{\partial \mu} + \frac{\partial u_0}{\partial \varkappa}\frac{d\varkappa}{d\mu} = \frac{\partial u_0}{\partial \mu} = \left[(\omega\mu)' \left(-\gamma \frac{\exp \alpha\tau(\mu)}{\operatorname{sh} \omega\tau(\mu)} + \beta \operatorname{cth} \varkappa(\mu) + \beta \frac{\alpha}{\omega}\right) + \right.$$

and

$$\left. + \omega\mu \left(-\gamma \frac{\exp \alpha\tau(\mu)}{\operatorname{sh} \omega\tau(\mu)} \varkappa(\mu) + \beta\right)\left(\frac{\alpha}{\omega}\right)'\right] \frac{2}{\beta - \gamma} \to +\infty$$

as $\mu \to 0$

$$\left(\lim_{\mu \to 0} (\omega\mu)' = h + \frac{1}{r}, \quad \lim_{\mu \to 0} \left(\frac{\alpha}{\omega}\right)' = -\frac{4}{r^2(\beta - \gamma)} > 0\right)$$

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STEADY ROTATIONAL-OSCILLATORY MOTIONS IN A SYSTEM WHOSE UNPERTURBED MOTION IS STABLE

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Method of consecutive approximations is used to construct a rotational-oscillatory solution of a general system with a parameter, and its stability is studied on the basis of the well-known theorems of the First Liapunov Method. Earlier, analogously stated problems were investigated in connection with the periodic or oscillatory solutions in the system with small parameters.

We investigate a system whose general form is

$$dx_i / dt = F_i(t, x_1, \ldots, x_n, \lambda) \qquad (i = 1, \ldots, n) \tag{1}$$

where $t \in [t_0, \infty)$ is an independent variable and $\lambda \in [\lambda_1, \lambda_2]$ is a numerical parameter whose value is, in general, not small. We assume that real functions F_i satisfy the following conditions.

1) Functions F_i are defined for all $t \in [t_0, \infty)$, continuous and T-periodic where T is constant and independent of λ.

2) Functions F_i are periodic in $x_1, \ldots, x_p$ $(0 \leqslant p \leqslant n)$ with periods $T_1, \ldots, T_p$, respectively, the latter also independent of λ.

3) Functions F_i have partial derivatives of first and second order in $x_1, \ldots, x_p$ and λ satisfying the Lipschitz conditions, with constant independent of t in the vicinity of a point belonging to some region G of the variables x_i and $\lambda_0 \in [\lambda_1, \lambda_2]$ unbounded in the coordinates $x_1, \ldots, x_p$